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Dependence on p of the Best L^p Approximation Operator

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For a fixed dimensional subspace and a function f, both contained in the intersection of a family of L^p spaces, the best approximation operator τ_f may be considered as a function of p as well as of the objective function f. We study the best approximation operator considering f fixed and p variable. © 1987 Academic Press, Inc.

1. INTRODUCTION

Let $(X, \mathscr{B}, \mu)$ be a measure space and suppose that V is a finite dimensional subspace of $L^1(X, \mathscr{B}, \mu) \cap L^\infty(X, \mathscr{B}, \mu)$ and that $f \in \{L^1(X, \mathscr{B}, \mu) \cap L^\infty(X, \mathscr{B}, \mu)\} \setminus V$. For $1 < p < \infty$ define the best L^p approximation operator at p, $\tau_f(p)$ to be the unique best approximation to f from V using the L^p-norm. That is, $\tau_f(p) = v_p$ with $\|f - v_p\|_p = \min\{\|f - v\|_p : v \in V\}$, where $\|h\|_p = \{\int_X |h|^p \, d\mu\}^{1/p}$. That v_p exists and is unique follows from the fact that V is a finite dimensional subspace of a strictly convex normed linear space. For $p = 1$ and $p = \infty$ define $\tau_f(p)$ to be the set of best approximations to f in the corresponding L^p norm from V. Finally, define $N_f(p)$ by $N_f(p) = \|f - \tau_f(p)\|_p$ for $1 < p < \infty$ and $N_f(p) = \|f - h\|_p$, $h \in \tau_f(p)$ for $p = 1$ or $p = \infty$.

In this setting we shall study the dependence of the best L^p approximation operator, $\tau_f(p)$, and the distance function, $N_f(p)$, on p. The behavior of the best approximation operator in various settings has been studied by many authors. The first study of this sort was given by Freud

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[6], who showed that the best uniform approximation operator mapping $C[a, b]$ into a finite dimensional Haar subspace satisfies a local Lipschitz condition at each $f \in C[a, b]$. These results were extended by Newman and Shapiro in [14]. In [9, 16] uniform continuity and uniform Lipschitz properties of this operator as a function of the elements being approximated was considered. In [1, 7, 10] similar studies of this operator in a fixed L^p space were presented. Differentiability properties as a function of the element being approximated and the behavior of the Lipschitz constant as a function of the dimension of the approximating subspace were studied in [10, 11]. Finally, in [4] the dependence of the best approximation problem on the norm being used was considered. Thus, for $p \geqslant 2$, it was shown that the coefficients of the best L^p approximant to a fixed f are continuous and differentiable functions of p when approximating on a continuum. It was noted there and in [5] that this continuous dependence on p is precisely the sort of information that one needs in order to develop a practical implementation of the Polya algorithm [12, 15]. In fact, in our setting the Polya algorithm and similar results for L^1 [13] can be interpreted as statements about the continuity of $\tau_f(p)$ at $p = \infty$ and $p = 1$.

2. Main Results

We begin by establishing two facts about $N_f(p)$.

Theorem 1. *Assume that $\mu(X) < \infty$. Then, either $N_f(p)/(\mu(X))^{1/p}$ is a strictly increasing function of p for $1 < p < \infty$ or there exists $v \in V$ such that $\tau_f(p) = v$ for all $p \in (1, \infty)$ and $|f - v| = k$ a.e. on X with $N_f(p) = k[\mu(X)]^{1/p}$ for all $p \in (1, \infty)$.*

Proof. Suppose that $1 < p < q < \infty$. Then by the definition of $\tau_f(p)$ and Hölder's inequality

$$N_f(p) = \left(\int_X |f - \tau_f(p)|^p \, d\mu\right)^{1/p} \leqslant \left(\int_X |f - \tau_f(q)|^p \, du\right)^{1/p}$$
$$\leqslant (\mu(X))^{(q-p)/pq} \left(\int_X |f - \tau_f(q)|^q \, d\mu\right)^{1/q} = (\mu(X))^{1/p - 1/q} N_f(q).$$

This shows that $N_f(p)/(\mu(X))^{1/p}$ is an increasing function of p. Now equality can occur in the second inequality if and only if $|f - \tau_f(q)| = k$ a.e. for some $k \neq 0$ as $f \notin V$. If strict inequality holds for each pair p, q with $p < q$ then we are done. If on the other hand, equality holds for some p and q, $1 < p < q < \infty$, then we must have that $|f - \tau_f(q)| = k$ a.e. We shall show that in this case that $h = \tau_f(q)$ is also equal to $\tau_f(s)$ for all $s \in (1, \infty)$.

Now it is known [3] that $h = \tau_f(r)$, $1 < r < \infty$, if and only if

$$\int_X |f-h|^{r-1} \operatorname{sgn}(f-h)\, v\, d\mu = 0 \qquad \forall v \in V.$$

For $r = q$, we must therefore have that

$$\int_X k^{q-1} \operatorname{sgn}(f-h)\, v\, d\mu = 0 \qquad \forall v \in V.$$

Thus,

$$\int_X \operatorname{sgn}(f-h)\, v\, d\mu = 0$$

and

$$\int_X k^{s-1} \operatorname{sgn}(f-h)\, v\, d\mu = 0 \qquad \forall v \in V$$

and each $s \in (1, \infty)$. Therefore, by the characterization theorem for best L^s approximations from V stated above we have that $h \in V$ is the best L^s approximation to f for each $s \in (1, \infty)$. Finally,

$$N_f(s) = \left(\int_X |f-h|^s \, d\mu \right)^{1/s} = k[\mu(X)]^{1/s}$$

for all $s \in (1, \infty)$ in this case. ∎

Next, we show that $N_f(p)$ is locally Lipschitz. In this theorem for convenience we shall normalize the problem by assuming $\mu(X) = 1$.

THEOREM 2. *Assume* $\mu(X) = 1$. *Then, for* $p, q \in (1, \infty)$, $|p-q| \leqslant 1$ *there exists a constant* $K > 0$, K *independent of* p *and* q *such that* $|N_f(p) - N_f(q)| \leqslant K|p-q|$.

Proof. The finite dimensionality of V implies that there exists $M < \infty$ such that $\|\tau_f(q)\|_\infty \leqslant M$ for all $q \in (1, \infty)$. Suppose $1 < q < p < \infty$. Then,

$$\begin{aligned} 0 \leqslant N_f(p) - N_f(q) &\leqslant \|f - \tau_f(q)\|_\infty^{(p-q)/p} \|f - \tau_f(q)\|_q^{q/p} - \|f - \tau_f(q)\|_q \\ &= \left[\left(\frac{\|f - \tau_f(q)\|_\infty^{1/p}}{\|f - \tau_f(q)\|_q^{1/p}} \right)^{p-q} - 1 \right] \|f - \tau_f(q)\|_q \end{aligned}$$

showing that

$$N(p) - N(q) \leqslant \|f\|_\infty (B^{p-q} - 1)$$

where $B=((\|f\|_\infty+M)/N_f(1))^{1/p}$. Thus, we have that

$$|N(p)-N(q)|\leqslant(\|f\|_\infty B\ln B)|p-q|=K|p-q|$$

as desired. ∎

Note that although $N_f(p)$ is strictly increasing unless $\tau_f(p)=\tau_f(q)$ for all p and q, Theorem 1 does not imply τ_f is 1–1. The following examples illustrate this.

EXAMPLE 1. In [17, Vol. 2, p. 248] Rice gives an example due to Descloux [2] in which the Polya algorithm fails. Specifically, a continuous function f on $[-1,1]$ is given such that for $V=\{ax: a\in\mathbb{R}\}$ and $\tau_f(p)=a_p x$, a_p fails to converge as $p\to\infty$. In fact, it is shown that there exists a monotone sequence p_k such that $a_{2p}<-\frac{1}{4}$ and $a_{2p+1}>\frac{1}{4}$. Thus, $\tau_f(p)$ assumes every value in $(-\frac{1}{4},\frac{1}{4})$ infinitely often.

EXAMPLE 2. Consider $X=\{1,2,3,4,5\}$ and $\mu(\{i\})=1/5$, the counting measure so that $L^p(X,\mathscr{B},\mu)$ is simply $\mathbb{R}^5$ with the l^p norm. Set $\mathbf{f}=(1, 2, -4, (3+\sqrt{657})/6, (3-\sqrt{657})/6)$ and $V=\{\alpha(1,1,1,1,1)\colon \alpha$ is real$\}$. Then it is easily seen that $\mathbf{O}=\tau_f(2)=\tau_f(4)\neq\tau_f(3)$.

EXAMPLE 3. Consider $\mathbb{R}^6$ with the l^p norm. Set $\mathbf{f}=(0,0,0,2,-1,-1)$ and $V=\{\alpha(1,1,1,1,1,1)\colon \alpha$ is real$\}$. Then $\tau_f(1)=\tau_f(2)=\mathbf{O}$, $\tau_f(\frac{3}{2})=-0.03257654$ and $\tau_f(\infty)=\frac{1}{2}$.

Example 3 has an interesting implication for L^p approximation when $p\neq 1$, 2 or ∞. Specifically, for a given L^p approximation problem with $p=1$, 2 or ∞ there may be a direct method for calculating the best L^p approximation. However, for an L^p approximation problem with p not one of these three values all current computational methods are iterative and require a starting estimate for $\tau_f(p)$. For example, in computing a best $L^{3/2}$ approximation one might start with an initial estimate for $\tau_f(\frac{3}{2})$ of $\frac{1}{2}\tau_f(1)+\frac{1}{2}\tau_f(2)$. Unfortunately, Example 3 shows that this need not be a better starting value than either $\tau_f(1)$ or $\tau_f(2)$.

Our last comment about the function $N_f(p)$ is a conjecture. Namely, we conjecture that $N_f(p)$ is a concave function of p. That is, for $p,q\in(1,\infty)$ and $0\leqslant\lambda\leqslant 1$, $\lambda N_f(p)+(1-\lambda)N_f(q)\leqslant N_f(\lambda p+(1-\lambda)q)$.

We now turn to the best approximation operator $\tau_f(p)$. As shown in the examples above, $\tau_f(p)$ need not be 1–1. The main question that we shall consider now is how many times can $\tau_f(p)$ assume a fixed value. We first consider this question in $\mathbb{R}^N$. Thus, let $X=\{1,2,\ldots,N\}$ and $\mu(\{i\})=1/N$ be the counting measure so that $L^p(X,\mathscr{B},\mu)$ becomes $\mathbb{R}^N$ with the l^p norm. In this setting, we show that $\tau_f(p)$ is either constant or at most $N-1$ to 1. We will have need of the following [8, p. 9, (3.1)].

LEMMA. *If* $0<\lambda_1<\lambda_2<\cdots<\lambda_n$ *then for* $t\in[a,b]$, $V=\{v(t)=\sum_{i=1}^n a_i\lambda_i^t\colon a_1,\ldots,a_n$ *real*$\}$ *is an* n *dimensional Haar subspace of* $C[a,b]$.

Now, let V be a subspace of $\mathbb{R}^N$ with dimension $=n<N$ and fix $\mathbf{f}=(f_1,\ldots,f_N)\in\mathbb{R}^N\setminus V$. For fixed p, $p\in(1,\infty)$ write $\tau_f(p)=(\tau_1^{(p)},\ldots,\tau_N^{(p)})$, where $\tau_f(p)\in V$ and is the best l^p approximation to f from V. Furthermore, set $\mathbf{r}^p=\mathbf{f}-\tau_f(p)=(r_1^{(p)},\ldots,r_N^{(p)})$. Then,

THEOREM 3. *Suppose that for a fixed* $p\in(1,\infty)$, *the coordinates of* $(|r_1^{(p)}|,\ldots,|r_N^{(p)}|)$ *contain* $k\leqslant N$ *distinct nonzero values* $\lambda_1,\ldots,\lambda_k$ *ordered as* $0<\lambda_1<\cdots<\lambda_k$. *Then either* $\tau_f(q)$ *is equal to* $\tau_f(p)$ *for all* $q\in(1,\infty)$ *or else* $\tau_f(q)$ *can equal* $\tau_f(p)$ *for at most* $k-1$ *distinct values of* q *(including the case* $q=p$*).*

Proof. Fix $\mathbf{v}\in V$, $\mathbf{v}=(v_1,\ldots,v_N)$ and define

$$F(t)=\sum_{i=1}^N |r_i^{(p)}|^{t-1}\operatorname{sgn}(r_i^{(p)})\,v_i=\sum_{j=1}^k b_j\lambda_j^{t-1}$$

where we have rewritten this sum in terms of the nonzero values of $|r_i^{(p)}|$. Now suppose that $\tau_f(p)$ is not best for all $q\in(1,\infty)$. Then, at each $q\in(1,\infty)$ for which $\tau_f(p)$ is optimal, we must have $F(q)=0$. Assuming $\mathbf{v}$ has been chosen such that $F(t)\not\equiv 0$, which is possible since $\tau_f(p)$ is not best for all $q\in(1,\infty)$, we have that $F(t)$ has at most $k-1$ zeros in $(1,\infty)$ by the lemma. ∎

Note that Example 3 shows that this result is sharp. That is, observe that c^* is the best l^p approximation to f provided c^* minimizes $3|c|^p+2|1+c|^p+|2-c|^p$. Setting $c=-0.01$, consider $g(p)=3(0.01)^{p-1}-2(0.99)^{p-1}+(2.01)^{p-1}$. Since $g(1)>0$, $g(\frac{3}{2})<0$ and $q(2)>0$ we see that there exist p_1 and p_2 with $1<p_1<\frac{3}{2}<p_2<2$ for which $g(p_1)=g(p_2)=0$. Since $g(p)$ is essentially the derivative of the expression that c^* must minimize it follows that $(-0.01)(1,1,1,,1,1)$ is the best l^{p_1} and l^{p_2} approximation to $\mathbf{f}$ from V. Finally, note that $\mathbf{f}-\tau_f(p_1)$ has exactly three nonzero distinct absolute values in its coordinates.

Along the same lines one has

COROLLARY 1. *If the set* $\{|r_i^{(p)}|\}_{i=1}^N$ *contains only one distinct value then* $\tau_f(p)=\tau_f(q)$ *for all* $q\in(1,\infty)$.

Note that this is not a necessary condition, as shown by the following example.

EXAMPLE 4. Let $\mathbf{f}=(\frac{1}{2}, 1, \frac{1}{2}) \in \mathbb{R}^3$ and set $V=\operatorname{span}\{(-1,0,1)\}$. Then $\tau_f(p)=(0,0,0)$ for $p \in (1,\infty)$ and $\{|r_i^{(p)}|\}_{i=1}^3$ has two distinct values. Also, note that this example shows that even if $\tau_f(p)=h$ for all $p \in (1,\infty)$, there need not be a unique best l^1 or l^∞ approximation to $\mathbf{f}$.

We now wish to return to the general setting. Thus, let $1<p<\infty$, $f \in L^\infty(X, B, \mu)$, $V \subset L^1 \cap L^\infty$ be a finite dimensional subspace and let $f \in (L^1 \cap L^\infty)\setminus V$ with $\|f\|_\infty = 1$. Fix $h \in V$ and define for $1<p<\infty$

$$\phi_f(p)=\int_X |f|^{p-1} \operatorname{sgn}(f)\, h \, d\mu.$$

Observe that $f, h \in L^1 \cap L^\infty$ implies $|\phi_f(p)| \leqslant \|f\|_\infty^{p-1} \int_X |h| \, d\mu < \infty$ for each $p \in (1,\infty)$, so that $\phi_f(p)$ is a well-defined function of p.

Now it is easily seen that $\phi_f(p)$ is an analytic function of p for $p \in (1,\infty)$. To establish this one first observes that

$$\frac{d^k \phi_f(p)}{dp^k}=\int_{\operatorname{supp}(f)} |f|^{p-1}(\ln|f|)^k \operatorname{sgn}(f)\, h \, d\mu.$$

By considering the derivative of the function $s(t)=\ln^k t/t^\alpha$ for $t \in [1,\infty)$ and $\alpha>0$, one sees that $|s(t)| \leqslant (k/\alpha e)^k$ with equality occurring at $t=e^{k/\alpha}$ (actually, $0 \leqslant s(t) \leqslant (k/\alpha e)^k$). Thus, rewriting $|f(x)|^{p-1}(\ln|f(x)|)^k$ — $\ln^k(1/f(x))/(1/f(x))$ and recalling that $\|f\|_\infty=1$ we see that $||f(x)|^{p-1} \ln^k|f(x)|| \leqslant (k/(p-1)\,e)^k$ a.e. Hence,

$$\left|\int_{\operatorname{supp}(f)} |f|^{p-1}(\ln|f|)^k \operatorname{sgn}(f)\, h\, d\mu\right| \leqslant \left(\frac{k}{(p-1)\,e}\right)^k \int_X |h| \, d\mu$$

showing that $\phi_f^{(k)}(p)$ exists and is finite for each $p \in (1,\infty)$, k a positive integer.

Now, fix $p \in (1,\infty)$ and consider the Taylor polynomial expansion of $\phi_f(q)$, $q \in (1,\infty)$, of degree n and remainder term $R_n(q)$. Here we have that for some ξ between p and q that

$$\begin{aligned} |R_n(q)| &= \left|\frac{(p-q)^{n+1}}{(n+1)!} \int_{\operatorname{supp}(f)} |f|^{\xi-1}(\ln|f|)^{n+1} \operatorname{sgn}(f)\, h\, d\mu\right| \\ &\leqslant \frac{(n+1)^{n+1}}{(n+1)!\, e^{n+1}} \left|\frac{(p-q)^{n+1}}{(\xi-1)^{n+1}}\right|. \end{aligned}$$

By an asymptotic formula for the Gamma function [18, p. 253],

$$\frac{(n+1)^{n+1}}{(n+1)!\, e^{n+1}}=\frac{1}{\sqrt{2\pi(n+1)}\,(1+\mathcal{O}(1/(n+1)))}$$

so that we see that the Taylor series of $\phi_f(q)$ converges to $\phi_f(q)$ in the interval $[(p+1)/2, p+(p-1)]$. From this it follows that $\phi_f(p)$ is analytic for $p \in (1, \infty)$. From this observation one has immediately that

THEOREM 4. *If $\tau_f(p)$ equals the same value of V for an infinite set of $p \in (1, \infty)$ having an accumulation point in $(1, \infty)$ then $\tau_f(p)$ is identically this value in V for all p.*

Proof. Suppose $\{p_i\}_{i=1}^{\infty} \subset (1, \infty)$, $p_i \to p_0 \in (1, \infty)$ with $\tau_f(p_i) = 0 \in V$ for all i. Fix $h \in V$ and consider $\phi_f(p)$ for this h. Since $\phi_f(p)$ is analytic at p_0 and $\phi_f(p)$ vanishes at each p_i by the characterization of best L^p approximations, it follows that $\phi_f(p) \equiv 0$ for all $p \in (1, \infty)$ by analytic continuation. Thus,

$$\int_X |f|^{p-1} \operatorname{sgn}(f)\, h\, d\mu = 0$$

$\forall p \in (1, \infty)$ and $\forall h \in V$. Hence $0 \in V$ is the best L^p approximation to f for all $p \in (1, \infty)$.

Example 1 shows that the requirement that the accumulation point be in $(1, \infty)$ is needed in the above theorem. This is not the case in the discrete setting. Specifically, let $X = \{n\}_{n=1}^{\infty}$, $\mathscr{B} = \mathscr{P}(X)$ and $\mu(n) = 1$ for all n. Write $l^p(Z)$ for $L^p(X, \mathscr{B}, \mu)$. Let $f \in l^1(Z)$ be fixed with $\|f\|_\infty = 1$ and let $V \subset l^1(Z)$ be a finite dimensional subspace not containing f. Then we have

THEOREM 5. *Suppose $\tau_f(p_j) = 0$ for $\{p_j\}_{j=1}^{\infty}$ with $p_j \uparrow \infty$. Then $0 \in V$ is the best approximation to f for all $p \in (1, \infty)$.*

Proof. Let $\gamma_1 > \gamma_2 > \gamma_3 > \cdots$ denote the distinct values taken on by $\{|f_i|\}_{i=1}^{\infty}$. Now, by hypothesis, **O** being the best $l^{p_j}(Z)$ approximation to **f** from V implies that

$$\sum_{i=1}^{\infty} |f_i|^{p_j - 1} \sigma(f_i)\, h_i = 0 \qquad \forall \mathbf{h} = \{h_i\}_{i=1}^{\infty} \in V \tag{1}$$

and conversely, where we have written $\sigma(f_i)$ for $\operatorname{sgn}(f_i)$. Now, consider the sums for $\mathbf{h} \in V$ fixed

$$\sum_{|f_i| = \gamma_l} \sigma(f_i)\, h_i \qquad \text{for} \quad l = 1, 2, \ldots. \tag{2}$$

We claim that for each l, there exists J such that $j \geqslant J$ implies (2) sums to 0. Indeed, if not, let γ_k be the largest γ_l for which (2) is not zero on some subsequence of $\{p_j\}$. Then for $\mathbf{h} \in V$ and p_j sufficiently large we have that

$$0\frac{1}{\gamma_k^{p_j-1}}\sum_{i=1}^{\infty}|f_i|^{p_j-1}\sigma(f_i)\,h_i$$

$$=\frac{1}{\gamma_k^{p_j-1}}\sum_{|f_i|>\gamma_k}|f_i|^{p_j-1}\sigma(f_i)\,h_i+\sum_{|f_i|=\gamma_k}\sigma(f_i)\,h_i$$

$$+\frac{1}{\gamma_k^{p_j-1}}\sum_{|f_i|<\gamma_k}|f_i|^{p_j-1}\sigma(f_i)\,h_i$$

$$=0+\sum_{|f_i|=\gamma_k}\sigma(f_i)\,h_i+\frac{1}{\gamma_k^{p_j-1}}\sum_{|f_i|<\gamma_k}|f_i|^{p_j-1}\sigma(f_i)\,h_i.$$

Now observe that for $p_j>2$,

$$\left|\frac{1}{\gamma_k^{p_j-1}}\sum_{|f_i|<\gamma_k}|f_i|^{p_j-1}\sigma(f_i)\,h_i f_i\right|\leqslant\frac{1}{\gamma_k^{p_j-1}}\sum_{|f_i|<\gamma_k}|f_i|^{p_j-1}|h_i|$$

$$\leqslant\left(\frac{\gamma_{k+1}}{\gamma_k}\right)^{p_j-1}\|\mathbf{h}\|_1$$

$$\to 0 \qquad \text{as} \quad p_j\to\infty.$$

Hence, we have that $0=\sum_{|f_i|=\gamma_k}\sigma(f_i)\,h_i=\sum_{|f_i|=\gamma_k}|f_i|^{p_j-1}\sigma(f_i)\,h_i$ for all j sufficiently large.

However, if there exists p_j such that $\sum_{|f_i|=\gamma_l}|f_i|^{p_j-1}\sigma(f_i)\,h_i=0$ then $\sum_{|f_i|=\gamma_l}\sigma(f_i)\,h_i=0$. Thus, for any $p\in(1,\infty)$ one then has that $\sum_{|f_i|=\gamma_l}|f_i|^{p-1}\sigma(f_i)\,h_i=0$. Since for each l there exists p_j such that (2) is zero it follows that for each $l=1, 2,...$ and $p\in(1,\infty)$ we have that

$$\sum_{|f_i|=\gamma_l}|f_i|^{p-1}\sigma(f_i)\,h_i=0.$$

Hence,

$$\sum_{i=1}^{\infty}|f_i|^{p-1}\sigma(f_i)\,h_i=\sum_{l=1}^{\infty}\sum_{|f_i|=\gamma_l}|f_i|^{p-1}\sigma(f_i)\,h_i=0.$$

Thus, $\mathbf{O}$ is the best $l^p(Z)$ approximation to f for all $p\in(1,\infty)$. ∎

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